UDC 539.3:62-50

FORMULATION OF THE PROBLEM OF CONTACT BETWEEN SEVERAL DEFORMABLE BODIES AS A NONLINEAR PROGRAMMING PROBLEM

PMM Vol.42, № 3, 1978, pp. 466-474

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(Received June 3, 1977)

A new formulation is given for the problem of the contact between several deformable bodies, and its equivalence to a convex programming problem is proved. On the basis of known results, theorems about the existence and uniqueness of the solution are established for the case of linear elasticity theory and the deformation theory of plasticity without unloading. Constraints are set on the external actions assuring the existence of a solution in the absence of a clamped part of the boundary, and their mechanical interpretation is given.

This paper is a continuation and development of the investigation in [1]. The Signorini method [2] is used, which is substantially the principle of possible Lagrange displacements in the presence of unilateral constraints. Out of the large quantity of papers devoted to the Signorini problem and its extensions, we note [3, 6].

The formulation developed below permits the application of the numerical methods of optimization theory (nonlinear programming) to the solution of contact problems. The numerical solution of some specific contact problems is given in [7, 10]; let us note that the deformability of the stamp in [7], however, was taken into account on the basis of a Winkler model and its generalization (an analogous idea is used in [8]), while the method of solving problems in [9, 10] is the method of direct search without analysis of the existence and uniqueness conditions for the solution.

1. Formulation of the problem. Let there be several solid deformable bodies of finite dimensions occupying the domains $\Omega^1, \ldots, \Omega^M$ with the boundaries $S^1 = \partial\Omega^1, \ldots, S^M = \partial\Omega^M$. Let us assume that the bodies are in contact at points in the undeformed state, or along pieces of surfaces whose shape and size are determined from geometric considerations.

To simplify the notation everywhere below (where possible), we agree to omit the superscript when speaking about a specific body and the parameters referred to it; if it is a question of two bodies making contact, we shall omit the superscript governing the number of one of the bodies and will replace the number of the second body by a prime.

Thus, let the boundary S of one of the bodies consist of three pieces: $S = S_u \cup S_\sigma \cup S_c$. We shall assume conditions of classical type

$$u(r) = g(r), \quad r \in S_u \qquad (1.1)$$
$$\sigma_{ij}(u)\nu_j = P_i(r), \quad r \in S_\sigma$$

to be given on S_u and S_σ. Here r is the radius vector of points Ω relative to the common coordinate system for all bodies, $u(r)$ is the displacement vector of the point r, σ_{ij} (u) are the stress tensor components related to the vector $u = u(r)$ by using the equation of state whose form still has to be fixed, v_j are components of the unit vector normal to S external to Ω; $g(r)$ are displacements given on S_u which are assumed zero for simplicity below, p are surface forces given on S_σ, and the repeated Latin subscripts denote summation between one and $n = 2,3$.

Let us take the hypothesis that the shape and size of the ultimate zones of contact along which the body Ω can make contact with the other bodies, can be indicated from geometric and physical considerations, and let us denote the set of these zones by S_c. We emphasize that only the ultimate sizes of the contact zones are assumed known, the actual contact areas are to be determined during solution of the problem.

We also assume that $\partial S_c = S_c \cap S_\sigma$. This hypothesis is used in giving the mathematical foundation and is ordinarily satisfied in technical applications.

Let ρF denote the density of the volume forces acting on the body Ω, then the problem, in the general case, consists of determining the state of stress and strain of each of the bodies Ω when the system of these bodies is subjected to the effect of surface and volume forces with densities P^1, ρF^1, . . ., P^M, ρF^M, as well as the size and shape of the contact zones and the contact pressure distributed over these zones. In such a general formulation, the problem may not have a solution at all, and if a solution exists, then it can turn out to be nonunique. The existence and uniqueness theorems will be established below for the case of linear elasticity theory and for deformation theory of plasticity without unloading.

2. Analysis of the boundary conditions in the contact zone. Let us examine two bodies Ω and Ω' making contact, and let S_c, S_c' denote the ultimately possible contact zone. We shall assume the shapes of the surfaces S_c and S_c' not to differ very strongly (the more exact meaning of this hypothesis will be clarified below). The equations describing the surfaces S_c and S_c' are taken in the form

$$\Psi(r) = 0, \quad \Psi'(r') = 0 \tag{2.1}$$

where r, r' are the radii-vector of points of the surfaces S_c and S_c'.

Let us select the functions Ψ, Ψ' in such a manner that the conditions

$$\Psi(r) > 0, \quad \text{if} \quad r \notin \Omega, \ \Psi(r) < 0 \ \text{ for } \ r \in \Omega \tag{2.2}$$

would be satisfied (the conditions are analogous for the function Ψ').

Because of deformation of the bodies Ω and Ω' the surfaces S_c and S_c' vary. Let us prove that the distortion of the shape of the body boundary is determined in a first approximation by normal (along the normal) displacements of particles lying on the boundary. For definiteness we consider the body Ω. Let r_0 be the radius-vector of particles of S_c prior to deformation, and let r be the radius-vector of the same particles after deformation; then we have

$$r = r_0 + u(r_0) \tag{2.3}$$

It follows from (2.1) and (2.3) that

$$\Psi(r - u(r_0)) = 0 \tag{2.4}$$

Taking the same hypothesis about the smoothness of Ψ as in [1], we linearize (twice) the dependence (2.4) with respect to u

$$\varphi(r, r_0) - u_{\nu n}(r_0) = 0 \tag{2.5}$$

We have here introduced the notation

$$\varphi(r, r_0) = \Psi(r) / |\operatorname{grad} \Psi(r_0)|, \quad u_{\nu n} = u\nu$$
$$\nu(r) = \operatorname{grad} \Psi(r) / |\operatorname{grad} \Psi(r)| \tag{2.6}$$

The dependence (2.5) means that the shape of the deformed boundary is determined, in a first approximation, by the normal displacements of the particles on it.

We now obtain the boundary conditions on S_c, S_c' in a first approximation. Let r_0' be the radius-vector of points of S_c' prior to deformation; because of the deformation governed by the displacement field u', these points occupy the position

$$r^* = r_0' + u_{\nu n}'(r'_0)\nu'(r'_0) \equiv r_0' + u_{\nu b}'(r_0') \tag{2.7}$$

Let us drop a perpendicular to the surface S_c from the point r^*. Let r_{00} denote the radius-vector of the intersection of this perpendicular with S_c. The following inequality holds

$$u_{\nu n}(r_{00}) \leqslant \delta^* \equiv (r^* - r_{00})\,\nu(r_{00}) \tag{2.8}$$

Evidently $r_{00} = r_{00}(r_0', u_{\nu b}')$. The purpose of the subsequent reasoning is to linearize this dependence, as well as condition (2.8) with respect to the normal displacements and the magnitude of the gap $|r_0' - r_0|$, where r_0 is the radius vector of points of intersection of the perpendicular drawn from the point r_0' to the surface S_c, with the surface S_c'.

With the accuracy assumed, it is possible to replace r_{00} by r_0 in (2.8). Indeed, it is easy to compute that

$$(r^* - r_{00})\nu(r_{00}) - (r^* - r_0)\,\nu(r_0) = O(|r_0 - r_{00}|^2 +$$
$$(r_0 - r_{00})(r_0' - r_0 + u'_{\nu b}(r_0')) \tag{2.9}$$

by using the assumption about the boundedness of the second derivatives of Ψ.

Performing the manipulations mentioned, we obtain the inequality

$$(u'_{\nu b}(r_0') + r_0' - r_0)\,\nu(r_0) - u_{\nu n}(r_{00}) \geqslant 0 \tag{2.10}$$
$$\forall\, r_0 \in S_c, \quad r_0' = r_0'(r_0)$$

from condition (2.8).

Replacement of $u_{\nu n}(r_{00})$ by $u_{\nu n}(r_0)$ is valid only under an additional assumption about the smallness of the strain (the first derivatives of the displacement), which is valid in the case of geometrically linear theories to which we limit ourselves here.

The dependence $r_0'(r_0)$ is determined from the formula $r_0' = r_0 + t_0 \operatorname{grad} \Psi(r_0)$, where t_0 is found from the equation

$$\Psi'(r_0 + t \operatorname{grad} \Psi(r_0)) = 0 \tag{2.11}$$

Following the general method, let us replace (2.11) by a linearized equation in t, which when solved yields

$$r_0'(r_0) = r_0 - \varphi'(r_0, r_0)\,\nu(r_0)/(\nu'(r_0)\,\nu(r_0)) \tag{2.12}$$

Interchanging the roles of the bodies Ω and Ω' we obtain

$$(u_{\nu b}(r_0(r_0')) + r_0(r_0') - r'_0 - u_{\nu b}'(r_0'))\ \nu'(r_0') \geqslant 0 \tag{2.13}$$

Let us assume that the scalar product of the difference between the unit normal vectors to S_c and S_c' by the displacement vector and the vectors of the initial gap $r_0'(r_0) - r_0$, $r_0(r_0') - r_0'$ is of second order of smallness with respect to the normal displacements of points of the surfaces S_c and S_c' and the magnitude of the gap between the bodies, then conditions (2.10) and (2.13) reduce formally to one condition

$$u_{\nu n} + u'_{\nu n} \leqslant \delta \tag{2.14}$$

The first of the forms of condition (2.14) can be used in solving specific problems when the independent variable is $r_0 \in S_c$, $\delta = (r_0'(r_0) - r_0)\ \nu(r_0)$, or the second possible form of this condition can be used when the independent variable is $r_0' \in S_c'$, $\delta = (r_0(r_0') - r_0')\ \nu'(r_0')$. The difference between the results will be of second order of smallness with respect to the magnitude of the initial gap and the normal displacements of points of the boundary.

3. Reduction of the problem to a variational inequality. Let us first consider the case when the bodies are linearly elastic, and we write the relation between the stress and strain for the body Ω in the form

$$\sigma_{ij}(u) = a_{ijkh}\varepsilon_{kh}(u) \tag{3.1}$$

where a_{ijkh} is the tensor of the elastic moduli possessing ordinary properties of symmetry and positive definiteness and generally dependent on points of the domain Ω. The strain tensor components ε_{ij} are related to the displacement vector by the Cauchy formulas $\varepsilon_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i})$.

We shall assume that

$$\begin{gathered} a_{ijkh} \in L^\infty(\Omega), \quad \rho F \in L^2(\Omega) \\ P_i \in L^2(\Omega), \quad u_i \in L^2(\Omega), \quad \varepsilon_{ij}(u) \in L^2(\Omega) \end{gathered} \tag{3.2}$$

then all the manipulations performed above will be valid. Let us introduce the functional space

$$V_* = \{v \mid v = v(r),\ r \in \Omega;\ v(r) = 0,\ r \in S_u;\ v \in H^1(\Omega)\} \tag{3.3}$$

where H^1 is the Sobolev space of functions possessing generalized summable-square first derivatives.

Formulation of the problem in differential form is

$$-\frac{\partial}{\partial x_j}[a_{ijkh}\varepsilon_{kh}(u)] = \rho F_i \tag{3.4}$$

It is assumed that conditions (1.1) are satisfied, and moreover

$$u_{\nu n} + u'_{\nu n} < \delta \Rightarrow \sigma_{ij}(u)\,\nu_j = 0, \quad u_{\nu n} + u'_{\nu n} = \delta \Rightarrow \sigma_\nu \leqslant 0, \quad \sigma_T = 0 \tag{3.5}$$

$$\sigma_\nu(u(r_0)) = \sigma_\nu(u'(r(r_0)))$$

$$(\sigma_\nu = \sigma_{ij}\nu_i\nu_j, \quad (\sigma_T)_i = \sigma_{ij}\nu_j - \sigma_\nu\nu_i) \tag{3.6}$$

where $r_0 \in S_c^\alpha$, $r = r^\beta(r_0)$ is defined in conformity with (2.12), the superscript β depends on α and equals the number of that body Ω with which Ω^α is in contact on a given piece of the surface S_c.

Let us multiply each of the equations (3.4) by v_i, where v is an arbitrary element of the space V_*, let us add the results obtained, and let us integrate over the domain Ω. Applying the Gauss — Ostrogradskii formula and utilizing (1.1), we find

$$a_*(u, v) = L_*(v) + \int_{S_c} \sigma_{ij}(u) v_i \nu_j \, dS \tag{3.7}$$

$$a_*(u, v) = \int_\Omega a_{ijkh} \varepsilon_{kh}(u) \varepsilon_{ij}(v) \, d\Omega$$

$$L_*(v) = \int_\Omega \rho F v \, d\Omega + \int_{S_\sigma} P v \, dS$$

Let us set $v = u$ in (3.7) and let us subtract the expression obtained from (3.6); by using the notation introduced we obtain

$$a_*(u, v-u) = L_*(v-u) + \int_{S_c} \sigma_{ij}(u)(v_i - u_i) \nu_j \, dS \tag{3.8}$$

Let V denote the direct product of the spaces V_* introduced above, i.e.,

$$V = V_*^1 \otimes \ldots \otimes V_*^M$$

Let us introduce a subset of functions K in the space V by means of the formula (v is an arbitrary element of V)

$$K = \{v \mid v \in V;\ v_{\nu n}^\alpha + v_{\nu n}^\beta \leqslant \delta\} \tag{3.9}$$

The superscripts α and β determine the numbers of the bodies touching in pieces of their boundaries. Adding all the equations (3.8), we find (here and below the summation is over all numbers of bodies)

$$a(u, v-u) = L(v-u) + \sum \int_{S_c} \sigma_{ij}(u)(v_i - u_i)\nu_j \, dS \tag{3.10}$$

$$\left(a(u, v) = \sum a_*(u, v),\ L(v) = \sum L_*(v)\right)$$

Let us separate the last sum in (3.10) into pairs of terms, each of which corresponds to a pair of touching surfaces S_c and S_c', and let us examine one of these pairs in detail

$$\int_{S_c} \sigma_{ij}(u)(v_i - u_i)\nu_j \, dS + \int_{S_c'} \sigma_{ij}(u')(v_i' - u_i')\nu_j' \, dS \tag{3.11}$$

We note first of all that

$$\sigma_{ij}(u)(v_i - u_i)\nu_j = \sigma_\nu(u)(v_{\nu n} - u_{\nu n}) + \sigma_T(u)(v_T - u_T) \tag{3.12}$$

where $v_{\nu n}$ is the normal component of the vector v (the projection on the normal ν) v_T is the projection of v on the tangent plane. Because of conditions (3.5) and (3.6), the second term in (3.12) is zero. We now recall that $\nu \simeq -\nu'$, $\sigma_\nu = \sigma_\nu'$ by assumption, and that the difference between the surfaces can be neglected in calculating surface integrals in the geometrically linear theory. There results from all the above that the expression (3.11) has the form

$$\int_{S_c} \sigma_\nu (v_{\nu n} + v'_{\nu n} - u_{\nu n} - u'_{\nu n})\, dS \tag{3.13}$$

By assumption $u = (u^1 (r^1), \ldots, u^M (r^M))$ is the solution of the problem (3.4) −(3.6), therefore $u \in K$. Now, let $v = (v^1 (r^1) \ldots, v^M (r^M))$ in (3.10) and (3.12) belong to K, then

$$v_{\nu n} + v'_{\nu n} \leqslant \delta$$

and, therefore, the integrand in (3.13) is nonnegative because of (3.5) and (3.6).

Therefore, the solution of the initial problem (1.1), (3.4)−(3.6) satisfies the variational inequality resulting from (3.10) and the nonnegativity of the pair of terms (3.11)

$$a (u, v - u) \geqslant L (v - u), \quad \forall v \in K, u \in K \tag{3.14}$$

The following theorem holds: The solution of the variational inequality (3.14), if it exists and possesses second derivatives (although generalized), will satisfy all the equations and conditions (1.1), (3.4) − (3.6).

The derivation of (3.4), the second condition in (1.1) and (3.5) is essentially no different than the derivation of analogous conditions and equations in [6]. The new condition (3.6), as compared with the problem in [6], results from the inequality (3.14), equation (3.4) and conditions (1.1) and (3.5) for a suitable selection of the components of the element $v = (v^1, \ldots, v^M)$; We shall not present the procedure for deriving condition (3.6) because of the simplicity of these manipulations.

Remark. The whole reasoning holds with slight insignificant changes even for the case of the deformation theory of plasticity without unloading (the physically nonlinear theory of elasticity) in which the relation between the stress and strain has the form [11]

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - 2\mu\omega (e_u) e_{ij} \tag{3.15}$$

where λ and μ are Lamé parameters and the constraints on the function $\omega (e_u)$ are mentioned in [1,6].

In this case the variational inequality has the form

$$a (u, v - u) - Dj (u. v - u) \geqslant L (v - u) \tag{3.16}$$

$$Dj (u, v - u) = 2\mu \sum \int_\Omega \omega (e_u (u)) e_{ij} (v-u) e_{ij} (v)\, d\Omega$$

where $a (u, v)$ and $L (v)$ are determined, as before, by the formulas presented in parentheses in (3.10). The later D denotes the Gateaux derivative of the functional $j (v)$ at the point u in the direction $v - u$.

4. Reduction of the problem to minimization of the functional. First, we assume that all $S_u \neq \phi$, then the positive definiteness of the bilinear form $a (u, v)$ on V follows from the Korn inequality for each of the forms $a_* (u, v)$:

$$a (v, v) \geqslant c \| v \|^2, \quad \forall v \in V, c = \text{const} > 0$$

The form $L (v)$ is continuous on V in the constraints formulated above on ρF, P. The symmetry and continuity of the form $a (u, v)$ on V is obvious.

It can be verified that the set K defined by (3.9) is convex in V. The closedness of this set results from the Lions theorem on traces [12]. The following theorem, which is a corollary of more general results [13, 14], therefore holds:

Theorem 4.1. The solution of the variational inequality (3.14) is equivalent to the problem of minimizing the functional

$$J(v) = {}^1/_2\, a(v, v) - L(v)$$

in the subset K of the space V.

By using the results of [1, 6], it can be proved that the following holds:

Theorem 4.2. The solution of the variational inequality (3.16) is equivalent to minimizing the functional

$$J(v) = {}^1/_2\, a(v, v) - L(v) - j(v)$$

in the subset K of the space V.

Introducing the spaces adjacent to V in subsets of rigid displacements, as is done in [15], analogous theorems can be established even for the case when some or all of the bodies Ω have no clamped sections of the boundary. Difficulties occur in the realization of this procedure, which are associated with setting up additional constraints on $\rho F, P$. A simpler path (which will indeed be used) is direct utilization of the variational inequalities (3.14) and (3.16) and the theorems relative to these inequalities, which are established in [16].

5. Existence and uniqueness theorems. Let us first note that the existence and uniqueness of the solution in the conditions of Sect. 4 result from more general results which can be found on monographs [13, 14]. We consequently examine the case when for all $\alpha = 1, \ldots, M$ we have $S_u^\alpha = \varnothing$. We shall use the following theorem [16].

Lions — Stampacchia theorem. Let the following conditions be satisfied:

1) The bilinear form $a(u, v)$ in V is continuous

$$|a(u, v)| \leqslant c\|u\|\|v\|, \quad \forall u \in V, \quad v \in V \tag{5.1}$$

2) The norm in V is equivalent to the norm $p_0(v) + p_1(v)$, where $p_0(v)$ is the norm in V relative to which the space V is pre-Hilbertian, and $p_1(v)$ is a seminorm in V;

3) The space

$$Y = \{v \in V \mid p_1(v) = 0\} \tag{5.2}$$

is finite-dimensional.

4) There exists a constant c_1 such that

$$\inf_y p_0(v - y) \leqslant c_1 p_1(v) \tag{5.3}$$

5) The form $a(u, v)$ is semi-coercive, i.e.,

$$a(v, v) > c_2 p_1^2(v), \ \forall v \in V, \ c_2 = \text{const} > 0 \tag{5.4}$$

6) The set K is closed and convex in V and contains the point $\{0\}$ (the zero element of the space V);

7) The form $L(v)$ is linear and continuous in V, where

$$L(y) < 0, \quad \forall y \in Y \cap K \tag{5.5}$$

Then the variational inequality (3.14) has at least one solution.

Let us confirm that all the conditions of the theorem formulated are satisfied in the problem under investigation. Let us note that

$$\|v\|_V^2 = \sum \int_\Omega [v_i v_i + \varepsilon_{ij}(v)\,\varepsilon_{ij}(v)]\, d\Omega \tag{5.6}$$

can be selected as the norm in V.

The equivalence of the norm (5.6) to the direct-product norm used above follows from the Korn inequality which is valid for each of the bodies Ω. We set

$$p_0^2(v) = \sum \int_\Omega v_i v_i d\Omega, \quad p_1^2(v) = \sum \int_\Omega \varepsilon_{ij}(v)\,\varepsilon_{ij}(v)\, d\Omega \tag{5.7}$$

Evidently $p_0(v)$ is a norm in L_2; with respect to this norm H^1, and therefore, V is also pre-Hilbertian. The quadratic form $p_1^2(v)$ is a semi-norm in V since $\varepsilon_{ij}(y) = 0$, where y is the displacement of the body Ω as if it were absolutely rigid. The space Y defined by (5.2) is finite, since it is the direct product of finite spaces of the form

$$Y^\alpha = \{y^\alpha \mid y^\alpha = a + b \times r^\alpha, \quad a = \text{const},\ b = \text{const}\} \tag{5.8}$$

Compliance with the inequality (5.3) is verified by the method of assuming the opposite. The semi-coercivity of the form $a(u, v)$ in the sense of (5.4) is a corollary of the Korn inequality for each of the bodies Ω. Condition 6) of the theorem has been established above.

Let us examine the constraint (5.5) in more detail and let us clarify its physical meaning. Let us note that (5.5) actually has the form

$$L(y) < 0, \ \forall y \in Y \cap K \setminus G \tag{5.9}$$

where G is the displacement of the system of bodies $\Omega^1, \ldots, \Omega^M$ as a single rigid whole. In fact, the following equalities hold:

$$\sum \left[\int_\Omega \rho F d\Omega + \int_{S_\sigma} P dS \right] = 0 \tag{5.10}$$

$$\sum \left[\int_\Omega r \times \rho F d\Omega + \int_{S_\sigma} r \times P dS \right] = 0$$

which are the equilibrium conditions for the system of bodies $\{\Omega\}$ as a whole. Substituting the expression for y^α from (5.8) in place of $y = (y^1, \ldots, y^M)$ in (5.5), and hence using (5.10), we arrive at (5.9).

Now it is easy to give a mechanical treatment of the condition (5.9): the work of the external forces on the displacement of each of the bodies Ω as a rigid whole from the state in which this body is prior to deformation is strictly negative, i.e., the condition (5.9) is the stability condition for the system of bodies prior to deformation.

The state of the system of bodies $\{\Omega\}$ can naturally turn out to be unstable as a result of deformation. Assuming the existence of a solution (a set of solutions U) of the problem posed and defining the set Z by means of the formula

$$Z = \{y \mid y \in Y;\ u + y \in K,\ \forall u \in U\}$$

we find

$$L(y) \leqslant 0, \quad \forall y \in Z \setminus G \tag{5.11}$$

from the variational inequality (3.14) and condition (5.10).

Condition (5.11) means that after deformation the system of bodies $\{\Omega\}$ turned out to be in equilibrium which is stable or indifferent.

The sets $Y \cap K$ and Z do not generally agree since it does not at all follow that $y \in K$ from the condition $u + y \in K$.

Therefore, it can be said that the condition (5.9), which is a slight extension of the "strong Signorini hypothesis" [3,4], assumes the mutual position of the bodies $\{\Omega\}$ to be fixed prior to deformation; condition (5.11) fixes this mutual arrangement of the bodies $\{\Omega\}$ after deformation and is a consequence of the nonlinearity of the problem under consideration.

We conclude on the basis of the Lions — Stampacchia theorem that there exists at least one solution of the problem posed, and the difference of two solutions belongs to a set of displacements of the system of bodies as a rigid whole.

The results of Sect. 5 on the existence and uniqueness of the solution carry over without change to the case of the deformation theory of plasticity (3.15). It is necessary to use the more general Minty — Browder theorem on monotonic operators [17,18], compliance with whose conditions follows from the results elucidated in [6] and Sect. 5 of this paper, in place of the Lions — Stampacchia theorem.

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Translated by M. D. F.